Design of Robust Nonsquare Constrained Model-Predictive Control

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A model-predictive control (MPC) design methodology for processes with more manipulated inputs than outputs is developed. Essential features of the proposed approach are the following: the on-line optimization minimizes an objective function based on the l_2 norm; an end-condition equation is utilized; model uncertainty is considered as upper and lower bounds on the pulse-response-model coefficients; hard constraints on the input and move-size variables and soft constraints on the output variables are posed. A major difference between square and nonsquare MPC is that in the former the end-condition can be used directly, while in the latter a nonlinear programming problem needs to be solved during the design phase to select values for the input move suppression coefficients. This technique is illustrated through a number of simulations and application to a real industrial process.

Introduction

Model-predictive control (MPC) methodologies have been widely used in the chemical industry throughout the past decade. The philosophy of MPC techniques is based on the on-line use of an explicit model of the process, to predict and optimize the effect of future control moves (manipulated variable changes) on the controlled variables (outputs), over a finite horizon. Among the various MPC algorithms, dynamic matrix control (DMC) requires simple step- (or pulse-) response models that can easily represent time delays, nonminimum phase characteristics, and complex dynamics. Using the DMC methodology, at each sampling point k we select the manipulated variables over a future control horizon, such that the predicted response based on the process model has some desired characteristics. The selection of the manipulated variables is the result of an optimization problem where the objective function is based on the difference between the predicted output trajectory and the desired set point over a future prediction horizon. The disturbance during the prediction horizon is assumed to be constant and equal to the difference between the process output and the model prediction at the current time step.

MPC is the only methodology to handle constraints in a systematic way during the design and implementation of the controller. This is the primary reason for the success of MPC techniques in numerous applications in the chemical process industries. Consequently, the issues of robust stability and performance of *constrained* MPC systems are of great industrial interest. These issues have been studied by several researchers (Morari and Zafiriou, 1989; Zafiriou, 1990; Zafiriou and Marchal, 1991; Genceli, 1993; Genceli and Nikolaou, 1993; Michalska and Mayne, 1993; Muske and Rawlings, 1993; Rossiter et al., 1993; Rawlings and Muske, 1993; Zheng and Morari, 1993; de Oliveira and Morari, 1994; Genceli and Nikolaou, 1995; Vuthandam et al., 1995).

A class of multi-input, multi-output (MIMO) systems for which no rigorous results exist for robust stability and performance is nonsquare systems, where the number of process input variables is larger than the number of process output variables. In the past, common practice has been to fix enough inputs at so-called "design values" or use additional controlled variables to make the system *square* before designing the controller. However, use of additional outputs may increase significantly the condition number of the system and make it more sensitive to modeling error (Morari et al., 1985). On the other hand, fixing some of the input variables results in fewer variables that can be changed simultaneously to control the system (Reeves and Arkun, 1989). In addition to the preceding, the following reasons make the utilization of all available input variables highly desirable (Forbes et al., 1992; Jerome and Ray, 1992; Lin and Hsieh, 1993; Henson et al., 1995): (a) there are more degrees of freedom in the control actions that can produce a certain output value; (b) in addi-

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tion, asymptotic stability can be achieved, even if some input variables are saturated. Since many chemical engineering systems are naturally *nonsquare*, a robust MPC design algorithm for such systems that utilizes all the available input variables would be very useful in ensuring closed-loop properties. The challenge in the analysis and design of constrained MPC systems is due to the following facts:

- Constraints on process inputs and outputs make the controller, and consequently the entire closed-loop, nonlinear. Therefore, linear control theory is inadequate for the design of constrained MPC systems.
- No closed-loop expression can be written for the controller, because the control action is the result of an on-line optimization problem.
- Model inaccuracy, which is an inevitable result of system identification methods, must be considered.

The objective of this article is to develop a rigorous methodology for the design of nonsquare constrained MPC systems with guaranteed robust stability and good performance in the presence of process/model mismatch. Essential features in the formulation of the on-line optimization problem are:

- The on-line objective function is based on the l_2 norm (quadratic dynamic matrix control (QDMC)).
- A nonsquare end-condition equation requires that the projected output values at the end of the prediction horizon equal the desired set points.
- Model uncertainty is considered as upper and lower bounds on the model coefficients.
- Hard constraints on the input and move size variables and soft constraints on the output variables are posed.
- The process is assumed to be open-loop stable.

The robust stability conditions we develop in the sequel pose lower bounds on the move suppression weights of the on-line objective function. These bounds depend on the modeling error and the control, prediction, and output constraint horizon lengths. The robust stability conditions are counterparts of conditions developed for square constrained MPC systems (Genceli, 1993; Vuthandam et al., 1995). A major difference between square and nonsquare MPC is that in the former the end-condition can be used directly, while in the latter a nonlinear programming problem must be solved during the design phase in order to select values for the move-suppression coefficients.

The rest of the article is structured as follows: we first formulate the optimization problem solved on-line when the QDMC with end-condition (EQDMC) methodology is applied to nonsquare systems. We then present our robustness results and develop a methodology for tuning nonsquare EQDMC controllers, so that closed-loop robust stability is guaranteed. Finally, the proposed controller design method is demonstrated through application to a food-extrusion process. Several simulations and experiments show that the technique is very effective in dealing with model uncertainties and various process disturbances.

Formulation of Nonsquare MIMO QDMC

Assume that a multivariable process behaving as

$$y(k) = d(k) + \sum_{j=1}^{N} H_j u(k-j) \tag{1}$$

is modeled by a unit-pulse response model

$$y(k|k) = d(k|k) + \sum_{j=1}^{N} G_j u(k-j). \tag{2}$$

We also assume that the additive error

$$E_j = H_j - G_j, \qquad 1 \le j \le N \tag{3}$$

is bounded according to the following inequalities:

$$|E_{klj}| \le \overline{E}_{klj}, \qquad 1 \le k \le N, \qquad 1 \le l \le no, \qquad 1 \le j \le ni. \tag{4}$$

The full minimization problem solved on-line by the MPC controller at each sampling point k is

$$\min_{z(k+1), z(k+2), \ldots, z(k+q), \Delta v(k+1), \Delta v(k+2), \ldots, \Delta v(k+m)} J(k), \tag{5}$$

where $J(k)$ is

$$J(k) = \sum_{i=1}^{q} \|\Xi z(k+i)\|_2^2 + \sum_{i=1}^{p} \|\Theta(y(k+i|k) - y^{sp})\|_2^2 + \sum_{i=0}^{m} \|R_i^{1/2} \Delta v(k+i)\|_2^2 \tag{6}$$

subject of the following constraints.

Model based output prediction

$$y(k+i|k) = d(k+i|k) + \sum_{j=1}^{i} G_j v(k+i-j) + \sum_{j=i+1}^{N} G_j u(k+i-j). \tag{7}$$

Disturbance prediction

$$d(k+i|k) = d(k|k) = y(k) - \sum_{j=1}^{N} G_j u(k-j) = d(k) + \sum_{j=1}^{N} E_j u(k-j). \tag{8}$$

Input move constraints

$$\Delta u_{j_{\min}} \le \Delta v_j(k+i) \le \Delta u_{j_{\max}}, \quad 1 \le j \le ni, \quad 0 \le i \le m. \tag{9}$$

Input constraints

$$u_{j_{\min}} \le v_j(k+i) \le u_{j_{\max}}, \qquad 1 \le j \le ni, \qquad 0 \le i \le m. \tag{10}$$

Output constraints

$$y_{j_{min}} - z_j(k+i) \le y_j(k+i|k) \le y_{j_{max}} + z_j(k+i),$$
$$z_j(k+i) \ge 0, \quad 1 \le j \le no, \quad 1 \le i \le q. \quad (11)$$

Methodology

In order to develop conditions for robust closed-loop stability of *nonsquare* QDMC methodologies we are going to utilize Lyapunov's direct (second) method. For discrete-time systems according to this, if a scalar function $\Phi(\boldsymbol{x}_k)$ of the states of a system $\boldsymbol{x}_k$ can be found such that the following inequalities are satisfied (Rawlings and Muske, 1993):

$$\Phi(\boldsymbol{x}_k) > 0, \qquad \boldsymbol{x}_k \ne 0 \quad (12)$$

$$\Phi(\boldsymbol{x}_{k+1}) < \Phi(\boldsymbol{x}_k), \quad \boldsymbol{x}_k \ne 0, \quad (13)$$

then the system is guaranteed to be stable.

Genceli and Nikolaou (1993) have shown that the application of the *end-condition* in the classic DMC framework improves the robust stability characteristics of the closed-loop system. The *end-condition* is an equality constraint that guarantees that at steady state the values of the input variables at the end of the control horizon are such that the predicted values of the output variables are equal to the desired setpoints. For a *square* system the *end-condition* is described by the following equation:

$$\boldsymbol{v}(k+m+i) = \boldsymbol{G}^{-1}[\boldsymbol{y}^{sp} - \boldsymbol{d}(k|k)], \qquad i \ge 0. \quad (14)$$

Genceli and Nikolaou (1993) utilized the *end-condition* to apply Lyapunov theory to *constrained square* QDMC systems. The extension of this methodology to *nonsquare* systems is not trivial. The main difficulty arises from the *end-condition* equation, Eq. 14, where $\boldsymbol{G}^{-1}$ does not exist, since $\boldsymbol{G}$ is a *nonsquare* matrix. In order to apply the Lyapunov theory to *nonsquare* QDMC, we shall use the following approach:

(i) Modify the end-condition equation, Eq. 14, to

$$\boldsymbol{v}(k+m+i) = \boldsymbol{Q}[\boldsymbol{y}^{sp} - \boldsymbol{d}(k|k)], \qquad i \ge 0, \quad (15)$$

where $\boldsymbol{Q} \in \Re^{ni \times no}$ is an unknown matrix that must satisfy the following equation:

$$\boldsymbol{GQ} = \boldsymbol{I}. \quad (16)$$

Since the number of inputs is greater than that of outputs, there is an infinite number of matrices $\boldsymbol{Q}$ that satisfy the equality given in Eq. 16. Our objective is to find the matrix that results in the best possible performance. It will be shown later that this matrix is the solution of a nonlinear optimization problem.

(ii) Use the objective function of the on-line minimization problem as a possible Lyapunov function after we modify it as follows:

$$\Phi(\boldsymbol{x}_k) = I(k) = \|\Xi\hat{\boldsymbol{z}}(k)\|_2^2 + \sum_{i=1}^{q} \|\Xi \boldsymbol{z}(k+i)\|_2^2$$
$$+ \|\Theta(\boldsymbol{y}(k) - \boldsymbol{y}^{sp})\|_2^2 + \sum_{i=1}^{p} \|\Theta(\boldsymbol{y}(k+i|k) - \boldsymbol{y}^{sp})\|_2^2$$
$$+ \sum_{i=-N+1}^{-1} \|\boldsymbol{R}_i^{1/2}\Delta\boldsymbol{u}(k+i)\|_2^2$$
$$\times \sum_{i=0}^{m} + \|\boldsymbol{R}_i^{1/2}\Delta\boldsymbol{v}(k+i)\|_2^2 + f(k). \quad (17)$$

The additional summations do not change the solution of the optimization problem, since all these additional terms belong to the present or past time, and their values are known. Since all the terms are positive for nonzero state vectors, the preceding function satisfies the first Lyapunov inequality.

(iii) Find a suboptimal solution $\Phi^*(\boldsymbol{x}_{k+1})$ at time $k+1$. If we can satisfy the inequality:

$$\Phi(\boldsymbol{x}_k) - \Phi^*(\boldsymbol{x}_{k+1}) \ge 0, \quad (18)$$

then the second Lyapunov condition, Eq. 13, is guaranteed to hold and $\Phi(\boldsymbol{x}_k)$ is a Lyapunov function. Inequality 18 is satisfied if the move suppression coefficients take values above certain lower bounds that are functions of the pulse-response coefficients, the estimated modeling error, the prediction, control and output-constraint horizons, and the unknown matrix $\boldsymbol{Q}$.

(iv) Formulate and solve a nonlinear optimization problem, so that the move suppression coefficients are minimized with respect to the matrix $\boldsymbol{Q}$. In the next section we present a theorem that formulates the optimization problem for QDMC with EQDMC methodology. A summary of the proof is given in the Appendix.

Robust Stability of Closed-Loop Nonsquare MIMO EQDMC

Theorem. Assume that:

(i) Dynamics of the real MIMO process are given by Eq. 1.

(ii) The disturbance $\boldsymbol{d}$ is bounded and remains constant after some time M, so that the following inequalities are satisfied:

$$d_{l_{min}} \le d_l(k) \le d_{l_{max}}, |\Delta d_l(k)| \le \Delta d_{l_{max}}, \qquad 1 \le l \le no \quad (19)$$
$$\Delta d_{l_{max}} = 0, \qquad k > M\left[\Rightarrow \lim_{k\to\infty} d_l(k) = d_{l\infty}\right]. \quad (20)$$

Then the closed-loop multivariable EQDMC system described by Eqs. 5 to 11 and Eq. 15 is robustly stable with zero offset if the following conditions are satisfied:

(i) $\boldsymbol{G}$ has rank *no*.

(ii) The lengths of prediction, soft constraint, and control optimization horizons (p, q, and m) satisfy the inequalities:

$$q-1 \ge m+1 \ge \max\left(\frac{u_{j_{max}} - u_{j_{min}}}{\Delta u_{j_{max}}}, 1\right) - 1, \qquad 1 \le j \le ni \quad (21)$$

$$q-1 \ge m+1 \ge \max\left(\frac{u_{j_{max}} - u_{j_{min}}}{\Delta u_{j_{max}}}, 1\right) - 1, \quad 1 \le j \le ni. \quad (22)$$

Table 1. Input/Output Steady State

$u_{1,s}$	105.0
$u_{2,s}$	7.7
$u_{3,s}$	475
$y_{1,s}$	7.9
$y_{2,s}$	36.5

(iii) Process modeling error and disturbance uncertainty satisfy the following inequalities:

$$\sum_{l=1}^{ni} \max\{G_{i,l}u_{l_{\max}}, G_{i,l}u_{l_{\mathrm{mon}}}\} \geq y_i^{sp} - d_{\min} + \sum_{l=1}^{ni}\left[U_l \sum_{j=1}^{N} \bar{E}_{i,l,j}\right], \quad 1 \leq i \leq no \quad (23)$$

$$\sum_{l=1}^{ni} \min\{G_{i,l}u_{l_{\max}}, G_{i,l}u_{l_{\mathrm{mon}}}\} \leq y_i^{sp} - d_{\max} - \sum_{l=1}^{ni}\left[U_l \sum_{j=1}^{N} \bar{E}_{i,l,j}\right], \quad 1 \leq i \leq no, \quad (24)$$

where $U_l = \max\{|u_{l_{\max}}|, |u_{l_{\min}}|\}$.

(iv) The move suppression terms $\boldsymbol{R}_m$ are the solutions of the following minimization problem:

$$\min_{Q} \sum_{i=1}^{ni} \boldsymbol{R}_m(i,i) \quad (25)$$

subject to the constraints

$$\boldsymbol{GQ} = \boldsymbol{I} \quad (26)$$

$$\frac{\min_l(\Delta u_{l_{\max}})}{\|\boldsymbol{Q}\|_\infty} - \max_l\left(\Delta u_{l_{\max}} \sum_{j=1}^{N} \|\bar{\boldsymbol{E}}_j\|_\infty\right) \geq \max_i(\Delta d_{i_{\max}}) > 0, \quad 1 \leq l \leq ni, \quad 1 \leq i \leq no, \quad (27)$$

Table 3. Condition Numbers

System	$\sigma_{\max}$	$\sigma_{\min}$	$\gamma(0)$
$u_1 - u_2 - u_3/y_1 - y_2 - y_3$	2.1960	0.0663	33.1135
$u_1 - u_2 - u_3/y_1 - y_2$	2.0515	0.5221	3.9293
$u_1 - u_2/y_1 - y_2$	1.8317	0.1838	9.9681
$u_1 - u_3/y_1 - y_2$	1.5213	0.5173	2.9407
$u_2 - u_3/y_1 - y_2$	1.6871	0.3817	4.4204

where

$$\boldsymbol{R}_m\boldsymbol{1}^T = [\boldsymbol{I} - \boldsymbol{W}^T\boldsymbol{D}]^{-1}\left(\left[\sum_{j=-N+1}^{m}(\boldsymbol{c}_j^* + \hat{\boldsymbol{c}}_j^*)\right] + (\delta + \hat{\delta})\boldsymbol{1}\right) \quad (28)$$

$$\boldsymbol{W} = \sum_{i=1}^{N} |\boldsymbol{Q}\bar{\boldsymbol{E}}_i| \quad (29)$$

$$D(i,i) = \sum_{j=1}^{ni} W(i,j), \quad 1 \leq i \leq ni \quad (30)$$

$$\boldsymbol{c}_j^*(k) = \sum_{l=1}^{ni} \sum_{i=-N+1}^{m} |C_{j,i}(k,l)|, \quad 1 \leq k \leq ni, \quad -N+1 \leq j \leq m \quad (31)$$

$$\hat{\boldsymbol{c}}_j^*(k) = \sum_{l=1}^{ni} \sum_{i=-N+1}^{m} |\hat{C}_{j,i}(k,l)|, \quad 1 \leq k \leq ni, \quad -N+1 \leq j \leq m \quad (32)$$

$$C_{j,i} = 0.5\left(X_{j,i} + X_{i,j}^T\right), \quad -N+1 \leq j \leq m, \quad -N+1 \leq i \leq m \quad (33)$$

$$\hat{C}_{j,i} = 0.5\left(\hat{X}_{j,i} + \hat{X}_{i,j}^T\right), \quad -N+1 \leq j \leq m, \quad -N+1 \leq i \leq m \quad (34)$$

Table 2. Impulse-Response Coefficients

j	$u_1 \rightarrow y_1$	$u_1 \rightarrow y_2$	$u_2 \rightarrow y_1$	$u_2 \rightarrow y_2$	$u_1 \rightarrow y_3$	$u_3 \rightarrow y_2$
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.0000	0.0000	0.0000	0.0000	-2.6487×10^{-1}
3	0.0000	0.0000	0.0000	0.0000	-2.5855×10^{-2}	-1.6122×10^{-1}
4	0.0000	0.0000	2.0101×10^{-1}	2.3186×10^{-1}	-1.5230×10^{-2}	-8.9264×10^{-2}
5	0.0000	0.0000	1.1833×10^{-1}	2.4465×10^{-1}	-1.1953×10^{-2}	-4.0466×10^{-2}
6	0.0000	0.0000	1.9905×10^{-1}	1.3584×10^{-1}	-3.7100×10^{-2}	-2.3025×10^{-1}
7	-7.8333×10^{-2}	0.0000	7.9644×10^{-2}	3.01117×10^{-1}	-1.1157×10^{-2}	-1.0993×10^{-1}
8	-7.1621×10^{-2}	-5.4833×10^{-2}	1.8356×10^{-1}	3.1459×10^{-2}	-1.1511×10^{-2}	-2.6693×10^{-2}
9	-3.1058×10^{-1}	-5.013×10^{-2}	7.3450×10^{-2}	6.3235×10^{-3}	-3.9845×10^{-2}	-1.7312×10^{-3}
10	-1.0411×10^{-1}	-2.1741×10^{-1}	4.8356×10^{-2}	1.0128×10^{-2}	-4.9496×10^{-2}	-3.1410×10^{-2}
11	-1.2217×10^{-1}	-7.2876×10^{-2}	3.8305×10^{-2}	1.5445×10^{-2}	-1.4875×10^{-2}	-9.4241×10^{-3}
12	-2.9051×10^{-1}	-8.5516×10^{-2}	-3.5271×10^{-3}	2.3800×10^{-2}	-5.6491×10^{-2}	-2.6292×10^{-2}
13	-1.1619×10^{-1}	-2.0336×10^{-1}	-1.4538×10^{-3}	-7.2793×10^{-3}	-9.0315×10^{-3}	-7.3899×10^{-3}
14	4.2154×10^{-2}	-8.1334×10^{-2}	4.3678×10^{-4}	-7.2160×10^{-2}	-1.3901×10^{-2}	-1.2778×10^{-2}
15	5.5713×10^{-3}	2.9508×10^{-2}	1.1135×10^{-2}	9.4949×10^{-4}	-6.9950×10^{-3}	-1.7983×10^{-2}
16	4.2489×10^{-2}	3.8999×10^{-2}	0.0000	0.0000	-1.7000×10^{-3}	-6.7624×10^{-3}
17	1.6982×10^{-2}	2.9743×10^{-2}	0.0000	0.0000	-1.3459×10^{-3}	4.7932×10^{-2}
18	-1.3358×10^{-2}	1.1888×10^{-2}	0.0000	0.0000	2.1251×10^{-3}	1.9151×10^{-2}
19	0.0000	-9.3503×10^{-3}	0.0000	0.0000	0.0000	-2.0125×10^{-2}
20	0.0000	0.0000	0.0000	0.0000	0.0000	-1.2767×10^{-2}

Table 4. Move-Suppression Coefficients R_j for 10% Uncertainty Level

i/j	0	1	2	3	4	5	6	7	8	9	10
(a) $u_1-u_2-u_3/y_1-y_2$ System											
1	13.8	14.1	14.4	14.7	15.0	15.4	15.8	16.1	16.5	16.9	17.3
2	11.9	12.1	12.4	12.7	13.1	13.5	13.9	14.3	14.7	15.1	15.5
3	10.0	10.2	10.4	10.7	11.0	11.3	11.6	12.0	12.3	12.7	13.0
(b) u_1-u_3/y_1-y_2 System											
1	14.0	14.2	14.5	14.9	15.2	15.5	15.9	16.3	16.7	17.1	17.5
3	10.0	10.2	10.4	10.7	11.0	11.3	11.7	12.0	12.3	12.7	13.1
(c) u_2-u_3/y_1-y_2 System											
2	14.1	14.3	14.6	15.0	15.3	15.6	16.0	16.4	16.7	17.1	17.5
3	10.2	10.4	10.7	10.9	11.2	11.5	11.8	12.2	12.5	12.9	13.2

$$X_{j,i}=\sum_{l=0}^{m}\bar{E}_{1-j}^{T}\Theta^{T}\Theta(\bar{E}_{1-i}-2S_{l,i})+\sum_{l=m+1}^{p-1}\bar{E}_{1-j}^{T}F_l^{T}\Theta^{T}\Theta F_l\bar{E}_{1-i}$$
$$-2\sum_{l=m+1}^{p-1}S_{l,i}^{T}\Theta^{T}\Theta F_l\bar{E}_{1-j}+\left(F_p\bar{E}_{1-j}-S_{p,j}\right)^{T}\Theta^{T}\Theta$$
$$\times\left(F_p\bar{E}_{1-i}-S_{p,i}\right),$$
$$-N+1\le j\le m,\qquad -N+1\le i\le m \quad (35)$$

$$\hat{X}_{j,i}=\sum_{l=0}^{q}\bar{E}_{1-j}^{T}F_l^{T}\Xi^{T}\Xi F_l\bar{E}_{1-i}+2S_{l,i}^{T}\Xi^{T}\Xi F_l\bar{E}_{1-j},$$
$$-N+1\le j\le m,\qquad -N+1\le i\le m \quad (36)$$

$$F_l=\sum_{k=1+l-m}^{n+1}G_kQ,\qquad 1\le l\le p \quad (37)$$

$$S_{l,i}=\sum_{k=2+l-i}^{N+1}G_k,\qquad 1\le l\le p,\qquad -N+1\le i\le m, \quad (38)$$

where δ, $\hat{\delta}$ are small positive numbers;

$$\mathbf{1}^T=[1\quad 1\cdots 1]^T\in\Re^{ni};\qquad Q\in\Re^{ni\times no}$$

(v) The move suppression terms are calculated from the recursive formula

$$R_j\mathbf{1}^T=\left[R_{j-1}+|Q\bar{E}_{1-j}|^TR_m\sum_{i=-N+1}^{0}|Q\bar{E}_{1-i}|\right]\mathbf{1}^T+c_j^*+\hat{c}_j^*$$
$$+(\delta+\hat{\delta})\mathbf{1}^T,\qquad 1\le j\le m-1. \quad (39)$$

Remark 1. The preceding optimization problem posed by Eq. 25 with the constraints 26–27 is nonlinear. In order to solve it we need an initial guess for matrix Q. The initial guesses we suggest are the inverses of all $no\times no$ subsystems. The optimization problem is solved based on those initial values, and the solution that gives the lowest value of the objective function is selected.

Robust nonsquare EQDMC design algorithm

In this section we summarize the proposed robust design algorithm for nonsquare EQDMC systems. The target of the algorithm is to obtain lower bounds for the tuning parameters R_j that guarantee robust stability, when the EQDMC methodology is used:

Step 1. Calculate $S_{l,i}$ from Eq. (38).
Step 2. Make an initial guess for the matrix Q (see Remark 1).
Step 3. Calculate F_l from Eq. 37.
Step 4. Calculate $X_{j,i}$ and $\hat{X}_{j,i}$ from Eqs. 35 and 36.
Step 5. Calculate $C_{j,i}$ and $\hat{C}_{j,i}$ from Eqs. 33 and 34.
Step 6. Calculate c_j^* and $\hat{c}_j^*$ from Eqs. 31 and 32.
Step 7. Find W from Eq. 29.
Step 8. Calculate $D\triangleq\operatorname{diag}[D(i,i)]$ from Eq. 30.
Step 9. Calculate $R_m\mathbf{1}^T$ from Eq. 28.
Step 10. Obtain a new matrix Q according to the nonlinear programming algorithm used for the minimization Eq. 25.
Step 11. Go to step 3, unless the last Q satisfies the termination criterion of the nonlinear programming algorithm.
Step 12. Calculate R_j, $1\le j<m-1$ from Eq. 39.

Robust Performance of Closed-Loop Nonsquare EQDMC

Let us define the performance index P of a closed-loop EQDMC system as

$$P=\sum_{k=0}^{\infty}\|\Theta(y(k)-y^{sp})\|_2^2$$
$$+\sum_{k=0}^{\infty}\sum_{i=1}^{no}\Xi(i,i)^2\max(0,[y_i(k)-y_{i_{\max}}],[y_{i_{\min}}-y_i(k)])^2. \quad (40)$$

Table 5. Move-Suppression Coefficients R_j for 5% Uncertainty Level ($u_1-u_2-u_3/y_1-y_2$ System)

i/j	0	1	2	3	4	5	6	7	8	9	10
1	9.6	9.8	10.0	10.3	10.5	10.8	11.0	11.3	11.6	11.8	12.1
2	8.2	8.4	8.6	8.8	9.1	9.4	9.7	10.0	10.3	10.5	10.8
3	6.8	7.0	7.2	7.4	7.6	7.8	8.0	8.3	8.5	8.8	9.0

Table 6. Move-Suppression Coefficients R_j for 13.74% Uncertainty Level ($u_1 - u_2 - u_3/y_1 - y_2$ System)

i/j	0	1	2	3	4	5	6	7	8	9	10
1	16.4	16.7	17.1	17.5	17.9	18.3	18.7	19.2	19.6	20.1	20.5
2	14.3	14.6	14.9	15.3	15.7	16.1	16.6	17.1	17.5	18.0	18.5
3	12.1	12.4	12.6	12.9	13.3	13.6	14.0	14.4	14.8	15.2	15.6

Performance improves as P decreases. The following corollary can be proved (Genceli, 1993), as a direct consequence of the previous theorem:

Corollary. The closed-loop multivariable nonsquare EQDMC system described by Eqs. 5 and 6 and the constraints 7–11 and 15 achieves a closed-loop performance P, no worse than the initially calculated optimal objective function, that is,

$$P \leq \Phi(0) = \min_{z(1),z(2),\ldots,z(q),\Delta v(0),\Delta v(1),\ldots,\Delta v(m)} J(0), \quad (41)$$

where

$$\begin{aligned} J(0) = &\|\Theta(y(0) - y^{sp})\|_2^2 + \sum_{i=1}^{no} \Xi(i,i)^2 \max \\ &\times (0,[y_i(k) - y_{i_{\max}}],[y_{i_{\min}} - y_i(k)])^2 \\ &+ \sum_{j=1}^{p} \|\Theta(y(j|0) - y^{sp})\|_2^2 + \sum_{j=0}^{q} \sum_{i=1}^{no} \Xi(i,i)^2_{\max} \\ &\times (0,[y_i(j|0) - y_{i_{\max}}],[y_{i_{\min}} - y_i(j|0)])^2 \\ &+ \sum_{j=-N+1}^{-1} \|R_j^{1/2}\Delta u(j)\|_2^2 + \sum_{j=0}^{N} \|R_j^{1/2}\Delta v(j)\|_2^2. \end{aligned} \quad (42)$$

Case Study

The tuning methodology was applied to a food-extrusion processing unit with three manipulated inputs u_1, u_2, u_3 and two outputs y_1, y_2. Due to confidentiality reasons we cannot give further information about the input and output variables. The main disturbance in food-extrusion processes is that natural raw materials vary in composition. They may also vary in moisture content as a result of weather variations during production and storage. A closed-loop MPC system was designed for this process using our tuning methodology. The simulations and experimental results presented in this section show the effectiveness of the method and its usefulness to industry.

Table 7. Q_{opt} for Different Uncertainty Levels

Uncertainty Level	Q_{opt}
5%	$\begin{bmatrix} -0.512 & 0.386 \\ 0.926 & 0.001 \\ 1.283 & -1.2267 \end{bmatrix}$
10%	$\begin{bmatrix} -0.751 & 0.327 \\ 0.634 & -0.072 \\ 1.158 & -1.298 \end{bmatrix}$
13.74%	$\begin{bmatrix} -1.177 & 0.806 \\ 0.114 & 0.514 \\ 0.936 & -1.047 \end{bmatrix}$

Model structure

To identify the model structure (approximate the dead times and the truncation order N) and estimate the model parameters G_j for the process, step changes were introduced into the control signals for the input variables and the output variables were measured. The step signals were applied around a steady state that according to previous experiments produces product of good quality. The steady state is shown in Table 1.

Deviation variables were created by subtracting the steady state shown in Table 1 from the input/output data. The resulting input/output responses were used to calculate the unit pulse-response coefficients. The sampling time was 4 seconds. Finally the pulse-response coefficients were scaled so that the maximum entry of each column and row of the gain matrix is one. This is a heuristic procedure (Morari et al., 1985) that we followed in order to minimize the condition number of the gain matrix. The pulse response coefficients are shown in Table 2. The models consist of 20 past response model coefficients and show that there are dead times (nonminimum phase characteristics) in all output responses. The steady-state gain matrix of the system is

$$G_{2\times 3} = \begin{bmatrix} -1.0000 & 0.9483 & -0.3044 \\ -0.6998 & 1.0000 & -1.0000 \end{bmatrix}.$$

Discrete-time step-response models were built for each pair of input/output variables using a sampling time of 4 seconds. The lengths of the move size, prediction, and output horizons were chosen to be 10, 30, and 15, respectively. The constraints that must be satisfied on the on-line optimization problem are:

Input Move Constraints:

$$-2.0 \leq \Delta v_1(k+i) \leq 2.0, \qquad 0 \leq i \leq m$$
$$-2.0 \leq \Delta v_2(k+i) \leq -2.0, \qquad 0 \leq i \leq m$$
$$-2.0 \leq \Delta v_3(k+i) \leq 2.0, \qquad 0 \leq i \leq m.$$

Input Constraints:

$$-10.0 \leq v_1(k+i) \leq 10.0, \qquad 0 \leq i \leq m$$
$$-10.0 \leq v_2(k+i) \leq 10.0, \qquad 0 \leq i \leq m$$
$$-10.0 \leq v_3(k+i) \leq 10.0, \qquad 0 \leq i \leq m.$$

Table 8. Closed-Loop Performances

Simulation	Performances		
1	$u_1-u_2-u_3/y_1-y_2$ System	u_1-u_3/y_1-y_2 System	u_2-u_3/y_1-y_2 System
	2004.0	2636.2	∞
2	$u_1-u_2-u_3/y_1-y_2$ System	u_1-u_3/y_1-y_2 System	u_2-u_3/y_1-y_2 System
	221.83	265.94	∞
3	5% Uncertainty Level	10% Uncertainty Level	13.74% Uncertainty Level
	204.76	221.83	270.76
4		EQDMC	QDMC
		221.83	409.59

Output Constraints:

$$-10.0 \le y_1(k+i|k) \le 10.0, \qquad 1 \le i \le q$$

$$-10.0 \le y_2(k+i|k) \le 10.0, \qquad 1 \le i \le q.$$

Remark 2. If we follow the conventional approach and try to make the system *square*, we need to either augment the primary outputs by a secondary one so that we have a 3×3 system, or delete one of the three manipulated inputs to create a 2×2 system. The former approach is not desirable, because by using an additional output we increase the cost of the control system, since we augment the optimization problem and we need additional hardware (such as sensors). By deleting an input we decrease the degrees of freedom of the system. Moreover results have been reported (Morari et al., 1985) where the addition of output variables to nonsquare systems increases the condition number of the system and makes it more sensitive to modeling error. In order to check these results in our system we considered a third secondary output variable. Step response tests were conducted in order to estimate the gains between each one of the manipulated variables and the additional output. The gain matrix of the system is augmented as follows:

$$G_{3\times3} = \begin{bmatrix} -1.0000 & 0.9483 & -0.3044 \\ -0.6998 & 1.0000 & -1.0000 \\ 0.9723 & -0.6732 & -0.8673 \end{bmatrix}.$$

In Table 3 we present the maximum and minimum singular values and the condition numbers for the 3×3, the 3×2, and all the 2×2 systems. The results show that the addition of the third output variable increases the condition number of the system by an order of magnitude. On the contrary, deletion of one input variable in the best case slightly decreases the condition number of the system.

Remark 3. The method presented here sets a limit in the maximum value of relative modeling error that can be tolerated. The constraint 27 of the off-line nonlinear minimization problem can be rewritten as

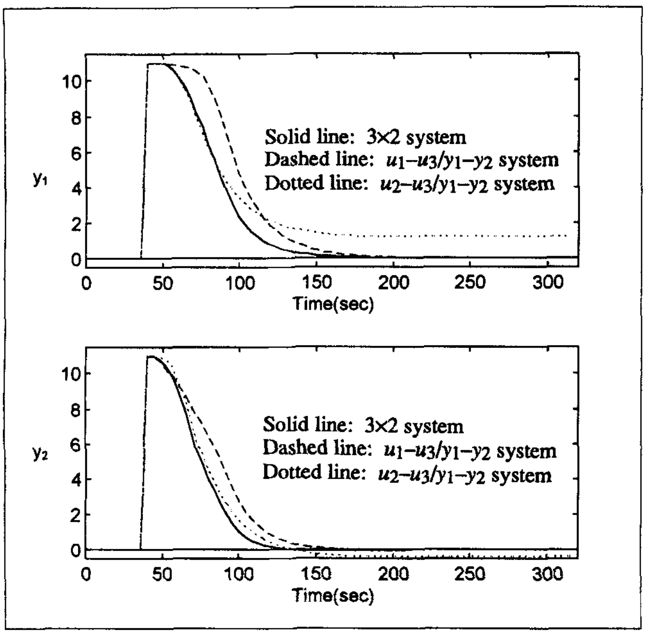

Figure 1. Step-rejection test, simulation 1 (outputs).

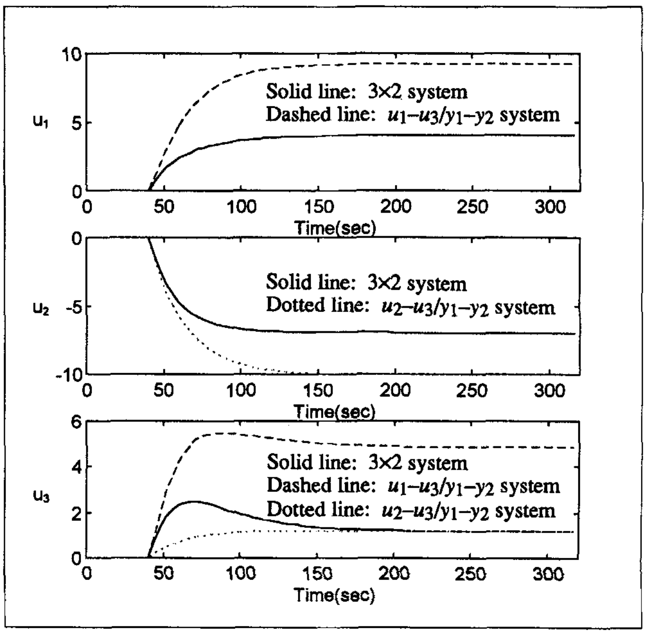

Figure 2. Step-disturbance test, simulation 1 (inputs).

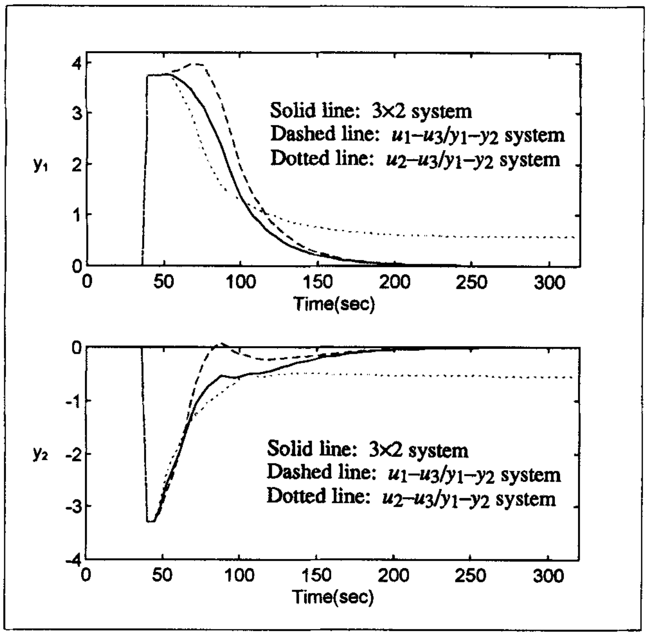

Figure 3. Step-rejection test, simulation 2 (outputs).

$$\sum_{j=1}^{N} \|\bar{E}_j\|_\infty \le \left(\frac{\min_l(\Delta u_{l_{\max}})}{\max_l(\Delta u_{l_{\max}})}\right)\cdot\frac{1}{\|Q\|_\infty},\qquad 1\le l\le ni,\qquad 1\le i\le no. \tag{42}$$

So the maximum tolerable relative error is obtained for the matrix Q with the minimum infinity norm. In our example the maximum tolerable relative error was found to be 13.74%.

Tuning of EQDMC controllers

The method was applied to the 3×2 system and to all 2×2 subsystems. Two implementations of the sequential quadratic programming (SQP) algorithm (subroutine DNCONF, IMSL, 1990; subroutine FSQP, Zhou and Tits, 1993) were used to solve the off-line nonlinear minimization problem. We considered 10% maximum relative error in each one of the pulse-response model coefficients. For the u_1-u_2/y_1-y_2 system, inequality 43 cannot be satisfied, so for that system we cannot find tuning parameters that guarantee the closed-loop robust stability. The move-suppression coefficients R_j were calculated for the 3×2 system and the remaining two 2×2 systems and are presented in Table 4. We also calculated the move-suppression coefficients for the 3×2 system when 5% and 13.74% maximum relative errors were considered. These values are shown in Tables 5 and 6. The opti-

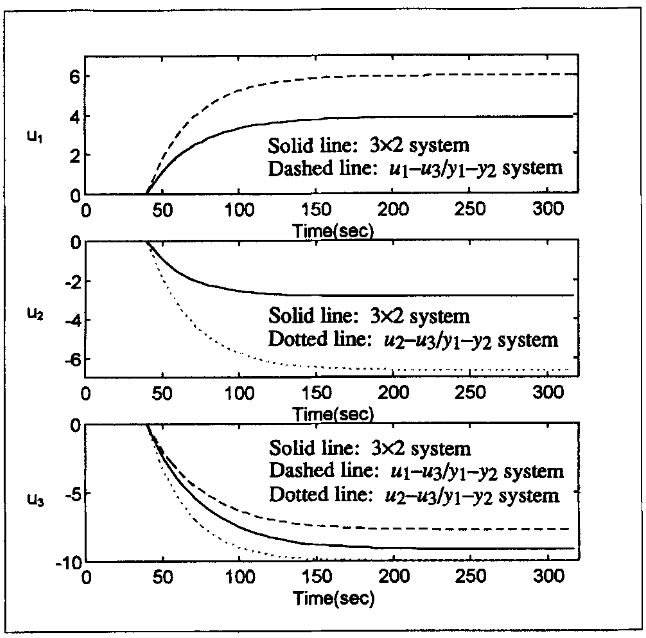

Figure 4. Step-rejection test, simulation 2 (inputs).

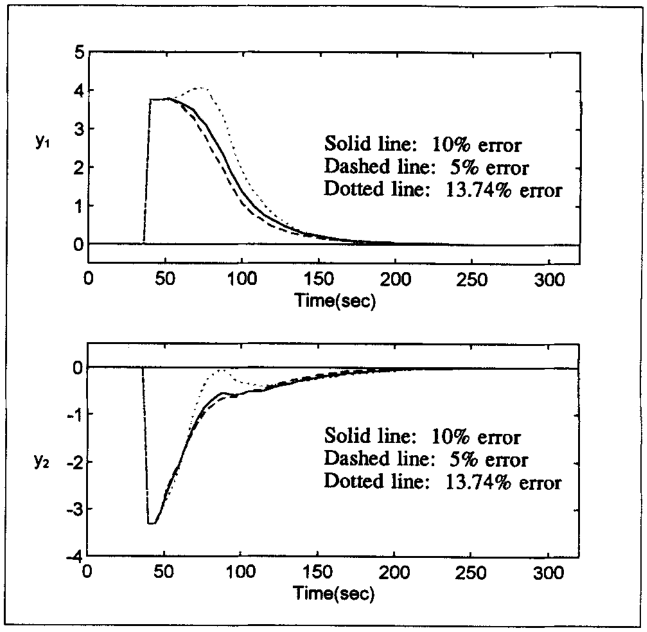

Figure 5. Step-rejection test, simulation 3 (outputs).

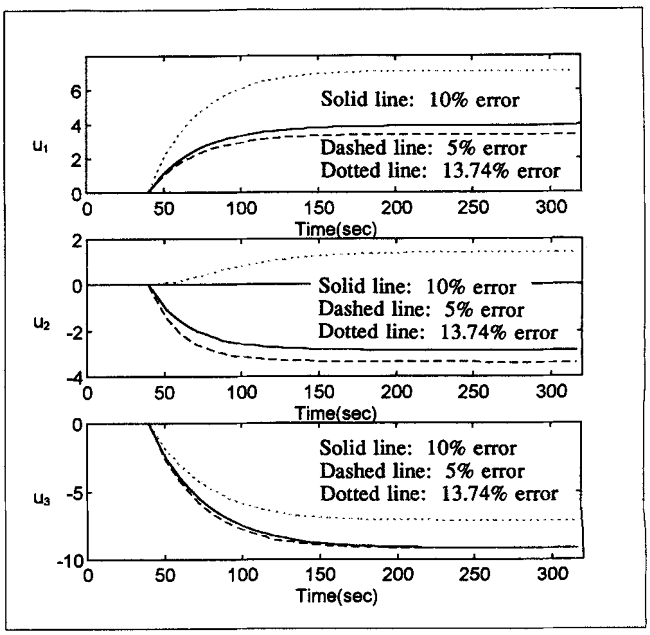

Figure 6. Step-rejection test, simulation 3 (inputs).

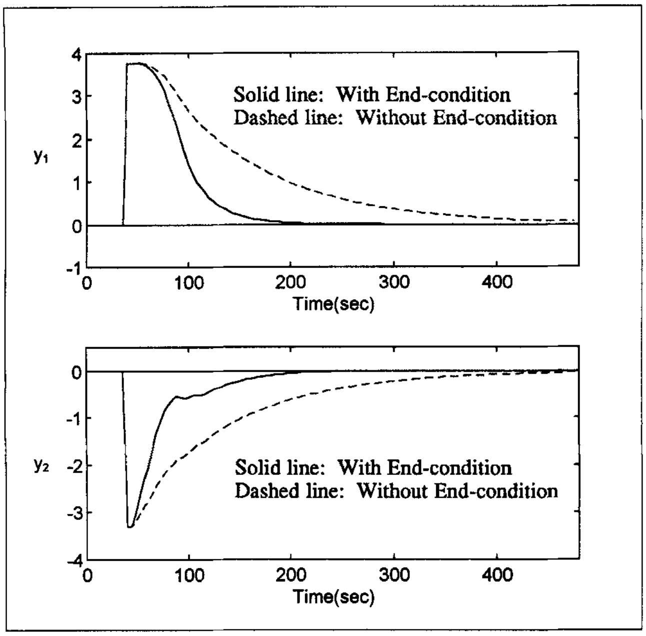

Figure 7. Step-rejection test, simulation 4 (outputs).

mum values for matrix Q for the different uncertainty levels are shown in Table 7.

Simulations

Using these tuning parameters, we applied the EQDMC methodology to all control configurations. We present here several disturbance rejection tests and compare the responses of the systems. The performances calculated by Eq. 41 are presented in Table 8.

Simulation 1. In this simulation we present the input and output responses (Figures 1 and 2) of the 3×2 system and the two 2×2 systems assuming 10% maximum relative error. The step disturbance

$$d_1 = 11.0, \qquad d_2 = 11.0$$

is introduced, causing temporary violation of output constraints.

Simulation 2. In this simulation we present the input and output responses (Figures 3 and 4) of the 3×2 system and the two 2×2 systems assuming 10% maximum relative error, when a step disturbance vector in the direction of the right singular vector of the gain matrix is introduced:

$$d_1 = 3.76, \qquad d_2 = -3.30.$$

Simulation 3. In this simulation we present the input and output responses (Figures 5 and 6) of the 3×2 system for different levels of maximum relative error (5%, 10%, 13.74%), when the step disturbance vector of simulation 2 is introduced.

Simulation 4. In this simulation we present the input and output responses (Figures 7 and 8) of the 3×2 system assuming 10% maximum relative error when EQDMC and QDMC algorithms are used and the step disturbance vector of simulation 2 is introduced. We use the same move-suppression coefficients for both methodologies.

Simulations showed that the robust performance of the nonsquare system is superior to the ones of the square systems. For both disturbance vectors, the system saturated and never achieved asymptotic stability. We also showed that the addition of the end-condition in the on-line formulation of the optimization problem improves the robustness characteristics of the closed-loop system.

Remark 4. There may be cases when the output disturbance is such that the end-condition cannot be satisfied, even

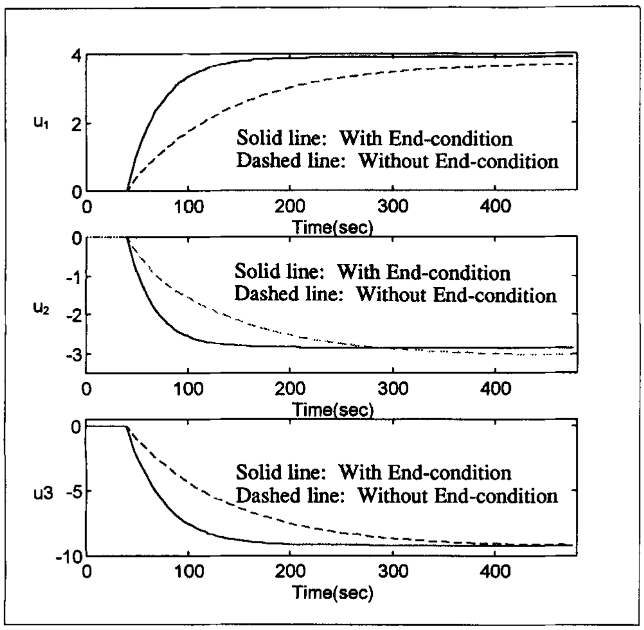

Figure 8. Step-rejection test, simulation 4 (inputs).

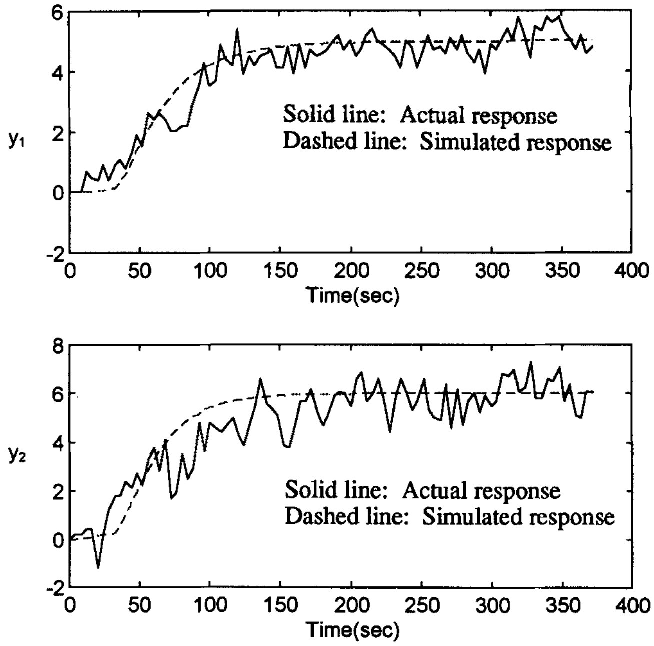

Figure 9. Setpoint step-change experiment.

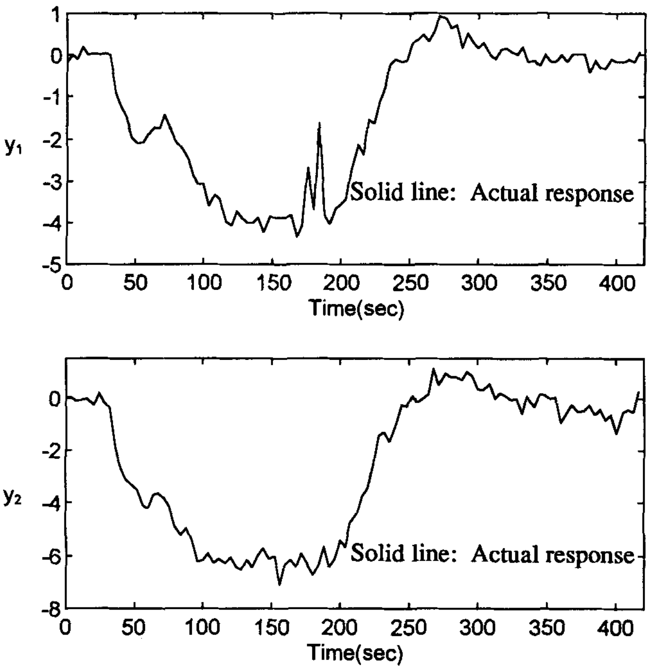

Figure 10. Disturbance rejection experiment.

after using the maximum input moves. In order to avoid situations like that, when we program the method we use the end-condition as an inequality constraint with a large weighting factor, rather than as an equality constraint.

Application to an industrial process

The EQDMC controller based on the 3×2 configuration succeeded in several tests, namely:

(1) Keeping the output variables at the initial set points.

(2) Rejecting step disturbances, including a change in the type of raw material used in the process (see Figure 9).

(3) Tracking setpoint changes in the output variables (see Figure 10).

The controller tracked any setpoint change and rejected any disturbance in less than 3 min (the initial target, based on the operators' experience was 15 min). The responses were very fast, although the robust stability conditions are only sufficient (and thus may be conservative). In Figure 10 we also plot the simulated responses to the same setpoint step-change test. It is clear that the real and predicted responses match well.

Conclusions

In this article we have presented rigorous sufficient conditions for robust closed-loop stability of EQDMC applied to processes with more input than output variables. The proposed controller design method needs simple pulse- or step-response models and upper and lower bounds on the error of the model parameters. The proposed controller structure (EQDMC) utilizes an end-condition, which not only is important in the development of a rigorous robust controller design methodology, but also improves closed-loop performance, when it is used as an additional constraint in the on-line implementation of the standard QDMC algorithm. The off-line EQDMC design involves the solution of a nonlinear optimization problem. Simulations and application of the method to a real industrial process elucidated the preceding issues and demonstrated the effectiveness of our EQDMC design algorithm.

Acknowledgment

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Notation

$d(k)\in\Re^{no}$ = actual disturbance vector at time k

$d(k+i|k)\in\Re^{no}$ = estimated disturbance vector at time $k+i$

$\bar{E}_j\in\Re^{no\times ni}$ = maximum modeling error, $1\le j\le N$

$G=\sum_{j=1}^{N}G_j\in\Re^{no\times ni}$ = the steady-state gain matrix of the model

$H_j\in\Re^{no\times ni}$ = jth unit pulse-response coefficients of the real plant

N = number of past input vectors used in the pulse-response model

$ni,\ no<ni$ = number of input and output variables

P = performance index

$R_j\in\Re^{ni\times ni}$ = input move-suppression diagonal-weight matrices for the jth input move in $J(k)$, $1\le j\le N$

$u(k)\in\Re^{ni}$ = actual input vectors at time k

$v(k+i)\in\Re^{ni}$ = calculated input vectors appearing in $J(k)$

$u_{j_{max}}$ and $u_{j_{min}}$ = upper and lower bounds on $v_j(k)$, $u_j(k)$, respectively, $1\le j\le ni$

$y_{j_{max}}$ and $y_{j_{min}}$ = upper and lower bounds on $y_j(k)$, respectively, $1\le j\le no$

$y(k)\in\Re^{no}$ = actual output vector at time k

$y(k+i|k)\in\Re^{no}$ = predicted output vector at time $k+i$, based on information at time k

$y^{sp}\in\Re^{no}$ = output setpoint vector

$z(k+1)\in\Re^{no}$ = constraint violation vector for $y(k)$

Norms

$\|A\|_\infty$ = matrix ∞-norm $\left(\max_i \sum_{j=1}^{n}|a_{i,j}|\right)$

$\|v\|_2$ = vector 2-norm $\left(\sum_{j=1}^{n} v_j^2\right)^{1/2}$

Greek letters

Δ = backward difference operator, $\Delta x(k)=x(k)-x(k-1)$

$\Delta u_{j_{max}}$ and $\Delta u_{j_{min}}$ = upper and lower bounds on $\Delta u_j(k)$, respectively, $1\le j\le ni$

$\Delta u(k)\in\Re^{ni}$ = input move vectors at time k

$\Delta v(k+i)\in\Re^{ni}$ = calculated input move vectors appearing in $J(k)$

$\Theta\in\Re^{no\times no}$ = diagonal-weight matrix for the output deviation term in $J(k)$

$\Xi\in\Re^{no\times no}$ = diagonal-weight matrix for the output constraint violation terms in $J(k)$

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Appendix: Theorem Proof

Assume that at time k the solution to the on-line optimization problem is the following set:

$$\{\hat{z}(k+1),\hat{z}(k+2),\ldots,\hat{z}(k+q),\Delta\hat{v}(k),\Delta\hat{v}(k+1),\ldots,\Delta\hat{v}(k+m)\} \quad \text{(A1)}$$

corresponding to the set of optimal predicted outputs

$$\{\hat{y}(k+1|k),\hat{y}(k+2|k),\ldots,\hat{y}(k+p|k)\}. \quad \text{(A2)}$$

This set also minimizes the augmented objective function, defined as

$$\Phi(x_k) = I(k) = \|\Xi\hat{z}(k)\|_2^2 + \sum_{i=1}^{q}\|\Xi z(k+i)\|_2^2 + \|\Theta(y(k)-y^{sp})\|_2^2 + \sum_{i=1}^{p}\|\Theta(y(k+i|k)-y^{sp})\|_2^2 + \sum_{i=-N+1}^{-1}\|R_i^{1/2}\Delta u(k+i)\|_2^2 + \sum_{i=0}^{m}\|R_i^{1/2}\Delta v(k+i)\|_2^2 + f(k) \quad \text{(A3)}$$

subject to the constraints 7–11 and 15, where:

$$\hat{z}_l(k) = \max\{[y_l(k)-y_{l_{\max}}],[y_{l_{\min}}-y_l(k)],0\} \quad \text{(A4)}$$

and $f(k)$ is an arbitrary function of k.

The minimum value of the objective function becomes:

$$\Phi(x_k) = I(k) = \|\Xi\hat{z}(k)\|_2^2 + \sum_{i=1}^{q}\|\Xi\hat{z}(k+i)\|_2^2 + \|\Theta(\hat{y}(k)-y^{sp})\|_2^2 + \sum_{i=1}^{p}\|\Theta(\hat{y}(k+i|k)-y^{sp})\|_2^2 + \sum_{i=-N+1}^{-1}\|R_i^{1/2}\Delta u(k+i)\|_2^2 + \sum_{i=0}^{m}\|R_i^{1/2}\Delta\hat{v}(k+i)\|_2^2 + f(k). \quad \text{(A5)}$$

The conditions (i)–(iii) of the theorem guarantee a feasible solution. Constraint 14 is an equality constraint. In the case of the square system this equation has a unique solution if the rank of the matrix is *no*. In the general case of nonsquare systems where $no \le ni$, and if the rank of G is *no*, there are infinite solutions. The minimization problem, Eq. 25, will find the solution that yields the smallest possible values of the move-suppression coefficients.

Assume that one solution of the equation is:

$$v(k+m) = Q[y^{sp} - d(k|k)]. \quad \text{(A6)}$$

From Eqs. 14 and A6 we obtain

$$GQ(y^{sp} - d(k|k)) = [y^{sp} - d(k|k)], \quad \text{(A7)}$$

which yields constraint 26.

From Eqs. 8 and A6 we have

$$v(k+m) = Q\left(y^{sp} - d(k) - \sum_{j=1}^{N} E_j u(k-j)\right). \quad \text{(A8)}$$

Now consider the augmented minimization problem at time $k+1$. A set of solutions that produce a feasible (but not necessarily optimal) solution, is the following:

$$\{\tilde{z}(k+2),\tilde{z}(k+3),\ldots,\tilde{z}(k+q+1),\Delta\tilde{v}(k+1),\Delta\tilde{v}(k+2),\ldots,\Delta\tilde{v}(k+m+1)\}$$
$$=\{\hat{z}(k+2)+|\hat{y}(k+2|k)-\tilde{y}(k+2|k+1),\ldots,\hat{z}(k+q)+|\hat{y}(k+q|k)-\tilde{y}(k+q|k+1)|,$$

$$|\tilde{y}(k+q+1)+y^{sp}|,$$
$$\Delta\hat{v}(k+1),\Delta\hat{v}(k+2),\ldots,\Delta\hat{v}(k+m),\Delta v(k+m+1)\}, \quad \text{(A9)}$$

where

$$\{\tilde{y}(k+2|k+1),\tilde{y}(k+3|k+1),\ldots,\tilde{y}(k+p+1|k+1)\}$$

is the corresponding set of predicted outputs.

In the set of A9 only $\Delta\tilde{v}(k+m+1)$ is to be selected. First the input vector $\tilde{v}(k+m+1)$ must be calculated, so that constraints 9, 10 and 14 are satisfied. Equivalently to Eq. A8 we get

$$v(k+m+1)=Q\left(y^{sp}-d(k+1)-\sum_{j=1}^{N}E_j u(k+1-j)\right), \quad \text{(A10)}$$

and from Eqs. A.8 and A.10,

$$\Delta v(k+m+1)=Q\left(y^{sp}-\Delta d(k+1)-\sum_{j=1}^{N}E_j\Delta u(k+1-j)\right). \quad \text{(A11)}$$

We can show that the set, Eq. A9, is feasible if inequality constraint 27 is satisfied. This yields a suboptimal cost $\Phi^*(k+1)$. The suboptimal cost $\Phi^*(k+1)$ satisfies the following inequality:

$$\Phi^*(k+1)\geq\Phi(k+1)\geq 0. \quad \text{(A12)}$$

We need to derive closed-loop robust stability conditions that guarantee that the sequence $(\Phi(k))_{k=0}^{\infty}$ converges. The sufficient conditions we need to develop should satisfy the inequalities

$$\Phi(k)\geq\Phi(k+1) \quad \text{(A13)}$$

or

$$\Phi(k)\geq\Phi(k+1)+[\Phi(k)-\Phi^*(k+1)]. \quad \text{(A14)}$$

After lengthy manipulations, inequality A14 yields

$$\Phi(k)\geq\Phi(k+1)+\|\Xi z(k)\|_2^2+\|\Theta(y(k)-y^{sp})\|_2^2$$
$$-(\beta+\hat{\beta})\Delta d_{\max}^2-\sum_{n=1}^{ni}\sum_{j=-N+1}^{m}\left(\delta+\hat{\delta}\right.$$
$$\left.+\sum_{l=1}^{ni}\sum_{i=-N+1}^{m}|C_{j,i}(k,l)|+|\hat{C}_{j,i}(k,l)|\right)\Delta\hat{u}_n(k+j)^2\sum_{n=1}^{ni}$$
$$\times\sum_{j=-N+1}^{m}\left(\sum_{l=1}^{ni}\sum_{i=-N+1}^{m}\left(\sum_{l1=1}^{ni}R_{l1,m}|Q\bar{E}_{1-j}|_{l1,n}|Q\bar{E}_{1-j}|_{l1,l}\right)\right)$$
$$\times\Delta\hat{u}_n(k+j)^2+\sum_{n=1}^{ni}\sum_{j=-N+1}^{m}(R_{n,j}-R_{n,j-1})\Delta\hat{u}_n(k+j)^2$$
$$+f(k)-f(k+1), \quad \text{(A15)}$$

where β and $\hat{\beta}$ are complicated functions of the variables to be calculated, but not important for the proof. If we can make the quantity added to $\Phi^*(k+1)$ in the righthand side of inequality A15 nonnegative, then we can guarantee that the sequence $\{\Phi(k)\}_{k=0}^{\infty}$ is nonincreasing. This can be achieved if $\{R_j\}_{j=-N+1}^{m}$ and $f(k)_{k=0}^{\infty}$ are chosen to satisfy the following inequalities:

$$\left(R_j-R_{j-1}-|Q\bar{E}_{1-j}|^T R_m\sum_{i=-N+1}^{0}|Q\bar{E}_{1-i}|\right)\mathbf{1}^T-c_j^*-\hat{c}_j^*$$
$$=(\delta+\hat{\delta})\mathbf{1}^T, \qquad -N+1\leq j\leq m \quad \text{(A16)}$$

$$f(k)-f(k+1)-(\beta+\hat{\beta})\Delta d_{\max}^2=\delta, \qquad 0\leq k\leq M \quad \text{(A17)}$$

$$f(k)=0, \qquad k>M. \quad \text{(A18)}$$

The solution of the system of equations is

$$R_m\mathbf{1}^T=[I-W^TD]^{-1}\left(\left[\sum_{j=-N+1}^{m}(c_j^*+\hat{c}_j^*)\right]+(\delta+\hat{\delta})I\right) \quad \text{(A19)}$$

$$R_j\mathbf{1}^T=\left[R_{j-1}+|Q\bar{E}_{1-j}|^T R_m\sum_{i=-N+1}^{0}Q\bar{E}_{1-i}|\right]\mathbf{1}^T$$
$$+c_j^*+\hat{c}_j^*+(\delta+\hat{\delta})I^T, \qquad -N+1\leq j<m \quad \text{(A20)}$$

$$f(k)=(\beta+\hat{\beta})\Delta d_{\max}^2(M+1-k), \qquad 0\leq k\leq M \quad \text{(A21)}$$

$$f(k)=0, \qquad k\succ M. \quad \text{(A22)}$$

If Eqs. A19–A22 are satisfied, then $\{\Phi(k)\}_{k=0}^{\infty}$ is a nonincreasing sequence and bounded below by zero. Therefore, it converges. This implies that taking the limit of inequality A15 as $k\to\infty$, yields

$$0\geq\lim_{k\to\infty}\left(\|\Xi z(k)\|_2^2+\|\Theta(y(k)-y^{sp})\|_2^2\right.$$
$$\left.+\sum_{j=-N+1}^{m}(\delta+\hat{\delta})\Delta\hat{v}(k+j)^2\right) \quad \text{(A23)}$$

$$\Rightarrow\lim_{k\to\infty}y(k)=y^{sp}. \quad \text{(A24)}$$

Q.E.D.

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